LIBRATION BOUNDARIES OF A TRIAXIAL SATELLITE IN A GRAVITATIONAL FIELD

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We shall consider the domains of possible and impossible oscillatory motions of a triaxial satellite in circular orbit in a gravitational field. The concept of an optimally stable satellite will be introduced. We shall show that such a satellite must have moments of inertia in the ratios 1.75:1:0.75.

The positions of the principal central axes of inertia $x_1x_2x_3$ of the satellite relative to the orbital tetrahedron $\boldsymbol{\tau}$, $\mathbf{n}$, $\mathbf{r}$ can be defined by means of the matrix of direction cosines

	x_1	x_2	x_3
$\boldsymbol{\tau}$	α_{11}	α_{12}	α_{13}
$\mathbf{n}$	α_{21}	α_{22}	α_{23}
$\mathbf{r}$	α_{31}	α_{32}	α_{33}

We assume that the orbit is circular. In this case $\boldsymbol{\tau}$ points in the direction of motion along the tangent to the orbit, $\mathbf{n}$ is directed along the normal to the orbital plane, and $\mathbf{r}$ along the radius vector of the orbit, Let A_1, A_2, A_3 be the principal central moments of inertia corresponding to the axes x_1, x_2, x_3; ω is the angular velocity of the center of mass of the satellite in orbit; p_1, p_2, p_3 are the components of the relative angular rotational velocity of the satellite along the axes x_1, x_2, x_3. Then (as was shown in [1]) there exists the first integral of the equations of motion

$$\tfrac{1}{2}(A_1p_1^2 + A_2p_2^2 + A_3p_3^2) + \tfrac{3}{2}\omega^2[(A_1 - A_3)\alpha_{31}^2 + (A_2 - A_3)\alpha_{32}^2] +$$
$$+ \tfrac{1}{2}\omega^2[(A_2 - A_1)\alpha_{21}^2 + (A_2 - A_3)\alpha_{23}^2] = h_0 \qquad (1)$$

and the equilibrium position

$$p_1 = p_2 = p_3 = 0, \quad \alpha_{11} = \alpha_{22} = \alpha_{33} = 1, \ \alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{23} = \alpha_{32} = \alpha_{31} = 0$$

is stable in the Liapunov sense if

$$A_2 > A_1 > A_3 \qquad (2)$$

In other words, in the equilibrium position the major axis of the inertial ellipsoid is directed along the radius vector and the minor axis along the normal to the orbital plane. Under condition (2) we can use (1) to construct the domains of possible and impossible satellite motions.

Since (1) is positively defined, we have the inequalities

$$3[(A_1 - A_3)\alpha_{31}^2 + (A_2 - A_3)\alpha_{32}^2] \leqslant 2h_0/\omega^2 \qquad (3)$$

$$(A_2 - A_1)\alpha_{21}^2 + (A_2 - A_3)\alpha_{23}^2 \leqslant 2h_0/\omega^2 \qquad (4)$$

$$3(A_1 - A_3)\alpha_{31}^2 + (A_2 - A_1)\alpha_{21}^2 \leqslant 2h_0/\omega^2 \qquad (5)$$

Let us consider (3), for example. Since α_{31}, α_{32}, α_{33} are the direction cosines of the radius vector $\mathbf{r}$ in the system x_1, x_2, x_3, it follows that inequality (3) together with the Eq.

$$\alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2 = 1 \qquad (6)$$

yields the domains of possible motion of the radius vector **r** on unit sphere (6). In this case the trihedron α_{31}, α_{32}, α_{33} is associated with the trihedron x_1, x_2, x_3 which coincides with it. Since cylindrical domains (3) describe curves symmetrical to the meridian $\alpha_{31}^2 + \alpha_{32}^2 = 1$ on sphere (6), it is sufficient to consider the projections of the domains of motion on the plane α_{31}, α_{32}; the shape, dimensions and disposition of domains (3) on this plane fully characterize the shape, dimensions, and disposition of domains (3) on sphere (6). In this plane model the point ($\alpha_{32} = 1$, $\alpha_{31} = 0$) is associated with the position of **r** along the minor axis x_2 of the inertial ellipsoid; the point ($\alpha_{32} = 0$, $\alpha_{31} = 1$) is associated with the position of **r** along the axis x_1 (the mean axis of the ellipsoid); finally, the point ($\alpha_{32} = 0$, $\alpha_{31} = 0$) is associated with the position of **r** along the major axis x_3 of the inertial ellipsoid of the satellite. Transformation of inequality (3) into an equation yields the boundaries of the libration domains.

Similarly, Expression (4) together with $\alpha_{21}^2 + \alpha_{23}^2 = 1$ yields the domains of motion of the normal **n** to the orbital plane relative to the principal central inertial axes x_1, x_2, x_3 of the satellite. Here the point ($\alpha_{23} = 1$, $\alpha_{21} = 0$) is associated with the coincidence of **n** with x_3; the point ($\alpha_{23} = 0$, $\alpha_{21} = 1$) with the coincidence of **n** with x_1; the point ($\alpha_{23} = \alpha_{21} = 0$) with the coincidence of **n** with x_2.

Finally, (5) together with $\alpha_{21}^2 + \alpha_{31}^2 = 1$ yields the domains of motion of the mean orbital system $\boldsymbol{\tau}$, **n**, **r**, The point ($\alpha_{21} = 1$, $\alpha_{31} = 0$) is associated with the coincidence of x_1 with **n**, the point ($\alpha_{21} = 0$, $\alpha_{31} = 1$) with the coincidence of x_1 with **r**, and the point ($\alpha_{21} = 0$, $\alpha_{31} = 0$) with the coincidence of x_1 with $\boldsymbol{\tau}$.

Simultaneous consideration of the above relations yields a sufficiently detailed notion of the boundaries of libration of an asymmetrical satellite. These boundaries can be represented as ellipses inscribed in unit circles,

$$\frac{\alpha_{31}^2}{a_r^2} + \frac{\alpha_{32}^2}{b_r^2} = 1, \quad \alpha_{31}^2 + \alpha_{32}^2 = 1, \quad a_r^2 = a_r'^2 h', \quad b_r^2 = b_r'^2 h' \tag{7}$$

$$a_r'^2 = \frac{\delta}{3(1-\varepsilon)}, \quad b_r'^2 = \frac{\delta}{3(\delta-\varepsilon)}, \quad \delta = \frac{A_2}{A_1}, \quad \varepsilon = \frac{A_3}{A_1}, \quad h' = \frac{2h_0}{A_2\omega^2}$$

$$\frac{\alpha_{21}^2}{a_n^2} + \frac{\alpha_{23}^2}{b_n^2} = 1, \quad \alpha_{21}^2 + \alpha_{23}^2 = 1 \quad a_n^2 = a_n'^2 h', \quad b_n^2 = b_n'^2 h' \tag{8}$$

$$a_n'^2 = \frac{\delta}{\delta - 1}, \quad b_n'^2 = \frac{\delta}{\delta - \varepsilon}$$

$$\frac{\alpha_{31}^2}{a_x^2} + \frac{\alpha_{21}^2}{b_x^2} = 1, \quad \alpha_{31}^2 + \alpha_{21}^2 = 1$$

$$a_x^2 = a_r^2, \quad b_x^2 = a_n^2 \tag{9}$$

We note that by the physical nature of the case and by virtue of (2) it is always the case that

$$\delta - \varepsilon > 1, \quad \varepsilon < 1, \quad \delta > 1 \tag{10}$$

For various values of h_0 (and fixed A_1, A_2, A_3) ellipses (7), (8) and (9) have various dimensions (Fig. 1). The domain of possible motion is the domain "inside" the corresponding ellipse (i.e. the domain containing the origin of coordinates). When (2) is fulfilled it is always the case that $a_r > b_r$, $a_n > b_n$, and $a_x > b_x$ if $A_2 - A_1 > 3(A_1 - A_3)$, and $a_x < b_x$ if $A_2 - A_1 < 3(A_1 - A_3)$.

The dimensions of the domains for fixed A_1, A_2, A_3 are determined by the value of h_0. For example, if these are certain initial deviations in the orbital plane which determine the value of h_0* and if the initial deviations from plane oscillations are arbitrarily small, the domain of possible transverse oscillations is still finite and is determined by the indicated value of h_0*.

For fixed h_0 the shapes and dimensions of the domains of possible motion are determined by the ratios of the moments of inertia.

Let us hold h' fixed and consider some examples.

Example 1. Let $\delta = 1.1$, $\varepsilon = 0.2$; then $a_r' = 0.68$, $b_r' = 0.64$, $a_n' = 3.32$, $b_n' = 1.11$, $a_x' = a_r'$, $b_x' = a_n'$.

Figure 1*a* shows the domains of possible motion for this case (the shaded areas correspond to the value $(h')^{1/2} = 0.1$). We note that the maximal possible deviations of the angles $(x_3 r)$, $(x_2 n)$, and $(x_1 \tau)$ are given by the relations

$$|\sin(x_3 r)| = a_r' \sqrt{h'}, \quad |\sin(x_2 n)| = a_n' \sqrt{h'}$$

$$|\sin(x_1 \tau)| = a_r' \sqrt{h'}, \quad \text{if} \quad A_2 - A_1 > 3(A_1 - A_3), \quad |\sin(x_1 \tau)| = a_n' \sqrt{h'}$$

$$\text{if} \quad A_2 - A_1 < 3(A_1 - A_3)$$

The angle $(x_3 r)$ characterizes pitching motion, and the angles $(x_2 n)$ and $(x_1 \tau)$ the lateral oscillations. From Example 1 (Fig. 1*a*) we see that quite moderate pitching oscillations can be accompanied by very considerable lateral oscillations. In fact, in this case we have $a_n'/a_r' = 4.899$, do that if the maximum pitch deviation is $(x_3 r) = 5°$, the roll deviation can reach $21°$.

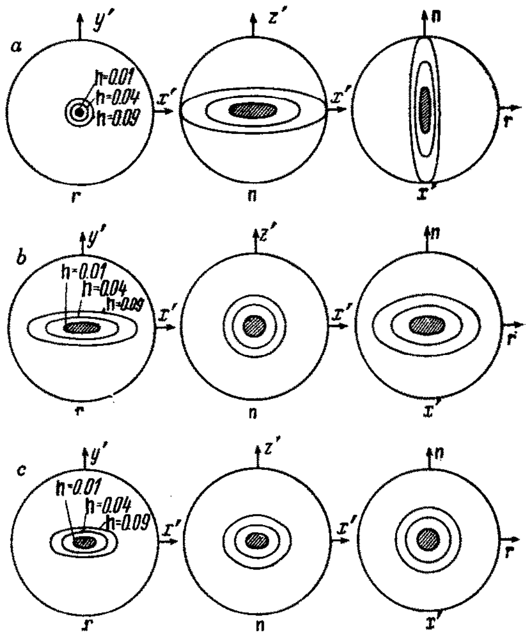

Fig. 1

The existence of a sufficiently broad domain of possible motion does not mean that the oscillations will necessarily build up to the extreme limits of this domain. All we are saying is that such buildup is possible with certain relations of the parameters. The actual oscillation buildup mechanism has to do with the commensurability of the spatial oscillations of the satellite. Kane [2] discerned the possibility of such buildup of transverse oscillations by numerical computations; Breakwell and Pringle [3] investigated these resonance effects by asymptotic methods.

Thus, domains of possible motion which are almost equal in size (for the same value of h') can be associated with both small oscillations lying deep inside the domain, and (in the case of a resonance relationship among the moments of inertial) with oscillations extending to the domain boundaries.

The existence of such resonance does not conflict with the previously proved [1] Liapunov stability relative to the equilibrium position of the satellite: in fact, in both the resonance and nonresonance cases we can always choose an h' so small that the oscillation amplitudes do not exceed a specified value.

Example 2. $\delta = 1.9$, $\varepsilon = 0.9$; then $a_r' = 2.52$, $b_r' = 0.795$, $a_n' = 1.45$, $b_n' = 1.38$, $a_x' = a_r'$, $b_x' = a_n'$.

In contrast to the above case, the moderate oscillations about the normal to the orbital plane are accompanied by more marked pitching oscillations and oscillations about the tangent to the orbit ($a_r'/a_n' = 1.75$; Fig. 1*b*).

We can now ask whether it is possible to choose moments of inertia of the satellite such that some uniform "moderateness" of all the satellite oscillations is guaranteed for a fixed h'. Let us call a satellite with the moment of inertia ratios ε^*, δ^* "optimally stable" if for a fixed value of the dimensionless energy constant h_n' these ratios minimize the maximal dimensions of the domain of possible motion. Since the maximal dimensions of the domain of possible motion are determined [1] either by a_r' or by a_n', the problem reduces to finding

$$\min_{\varepsilon,\delta} \max (a_r', a_n')$$

Let us consider domain (10) (Fig. 2) on the plane ε, δ. The straight line $\delta + 3\varepsilon = 4$ divides this domain into two parts. Above this line we have $a_r'^2 > a_n'^2$ and we must minimize a_r' as the larger of the two quantities a_r' and a_n'. Below the line we have $a_n'^2 > a_r'^2$ and the quantity to be minimized is a_n'. The straight lines $a_n'^2 = a_{n0}'^2 =$ const are equivalent to some straight lines $\delta =$ const, and the values of $a_n'^2$ diminish monotonously from ∞ as δ increases from 1. The smallest value of $a_n'^2$ in the subdomain $\delta + 3\varepsilon \leqslant 4$ is attained at the intersection of the boundary $\delta + 3\varepsilon = 4$ of this subdomain with the boundary $\delta - \varepsilon = 1$ of domain (10). Now let us consider the subdomain $\delta + 3\varepsilon \geqslant 4$. Here the quantity to be minimized is a_r'. The isolines $a_r'^2 = a_{r0}'^2 =$ const are the bunch of straight lines $\delta + 3a_{r0}'^2\varepsilon = 3a_{r0}'^2$ passing through the points $\delta = 0$, $\varepsilon = 1$, $\delta = 3a_{r0}'^2$, $\varepsilon = 0$, so that the minimal value of $a_r'^2$ in the subdomain $\delta + 3\varepsilon \geqslant 4$ corresponds once again to the intersection of the boundary $\delta + 3\varepsilon = 4$ of this subdomain with the boundary $\delta - \varepsilon = 1$ of domain (10). This intersection is associated with the values $\varepsilon = 0.75$, $\delta = 1.75$.

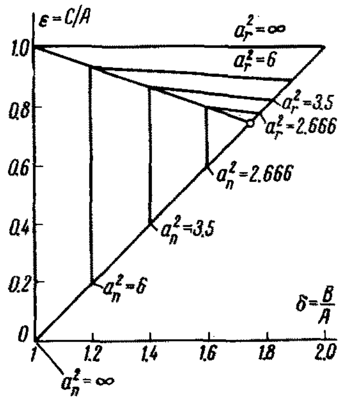

Fig. 2

Thus, the optimally stable satellite has its moments of inertia in the ratios

$$A_2^*: A_1^*: A_3^* = 1.75 : 1 : 0.75 \tag{11}$$

Example 3. Domains of possible motion of optimally stable satellite (11). In this case $a_r' = 1.526$, $b_r' = 0.7513$, $b_n' = 1.32$, $b_x' = a_n' = a_r' = a_x'$ (Fig. 1c).

Ratios of moments of inertia (11) are possible only with a degenerate disk-shaped satellite. In fact, one ought to consider a real satellite with moment ratios close to (11).

We note likewise that on an elliptical orbit the boundary $\delta - \varepsilon = 1$ is the resonance curve of parametric resonance in the lateral oscillations [4], so that point (11) is a resonance point. It also means that we must take not (11), but some relation (from the nonresonance zone) which is close to it.

It is noteworthy that the above resonance curve has a second branch not indicated in [4 and 5] but determined in [3]. No other curve of those found in [4, 6 and 3] passes through point (11).

In conclusion we note that small oscillations in a circular orbit of a satellite with parameters (11) are given by the system of Eqs. (with dimensionless time)

$$\alpha^{\cdot\cdot\prime\prime} + {}^3/_7\alpha'' = 0, \qquad \alpha^{\cdot\cdot\prime} + \alpha' = 0, \qquad \gamma^{\cdot\cdot\prime} + 4\gamma' = 0 \tag{12}$$

The pitching, rolling, and yawing oscillations (α'', α', and γ', respectively) are mutually independent (the system splits, as everywhere, along the straight line $\delta - \varepsilon = 1$). Integration of the system is elementary. The periods of the pitching, rolling and yawing oscillations are respectively equal to $\sqrt{7/3}$, one, and one half of the period of revolution of the satellite center of mass.

BIBLIOGRAPHY

1. Beletskii, V.V., On satellite libration. Collection of works "Artificial Satellites of the Earth", Issue No. 3, 1959.
2. Kane, T.R., Attitude Stability of Earth-pointing satellites. A.I.A.A.J. Vol. 3, No. 4, 1965.

3. Breakwell, J.V. and Pringle, R., Preprint of a paper presented at the 15th Congress of the International Astronautical Federation, Athens, 1965.
4. Beletskii, V.V., Libration of a satellite on an elliptical orbit. Collection of works "Artificial Satellites of the Earth", Issue No. 16, 1963.
5. Beletskii, V.V., Motion of an Artificial Satellite About Its Center of Mass, "Nauka", Moscow, 1965.
6. Zlatoustov, V.A., Okhotsimskii, D.E., Sarychev, V.A. and Torzhevskii, A.P., A study of satellite oscillations in the plane of an elliptical orbit. Kosmicheskie Issledovaniia Vol. 2, No. 5, 1964.

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